Limits of Rauzy graphs of low-complexity words

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August 7, 2019

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The Benjamini-Schramm limit

Limits of finite graphs

The aim of the thesis is to show the convergence properties of particular sequences of finite graphs. Before doing this, it is natural to ask what it really means to take a “limit” of finite graphs, and what problems one might encounter in defining this.

So to begin, we motivate the definition of the *Benjamini-Schramm limit* by highlighting the complication with graphs.

Limits of finite graphs

Graph limits: attempt #1

Consider the following sequence of graphs: take $B_n(\mathbb{Z}^2, 0)$ to be the ball of radius n around 0 in the integer lattice (with the graph metric). Imagine coloring one of these balls the red, and color a second such ball blue, and then connect the two at $(0, 0)$ with a line of length n. Call these graphs G_n.

Limits of finite graphs (Graph limits: attempt #1 (cont.))

This sequence G_n illustrates a particular problem. As $n \to \infty$, there are two growing graphs of different colors, which are moving away from eachother. Naively, in the "limit" one has two copies of $\mathbb{Z}^2$, with different colors, **infinitely far away from eachother**, with some line "connecting" them.

This is obviously a problem — the limit may be an infinite graph, but no two vertices should be infinitely far apart. What is it that goes wrong?

Limits of finite graphs (Graph limits: attempt #1 (cont.))

The issue is that the space of graphs simply cannot itself be given any reasonable topology. We are forced to think about *rooted* graphs.

The limit depends on how we choose the roots!

But, if one assigns the root into the red ball, then the blue ball "disappears off to infinity", and conversely if one assigns the roots into the blue ball, then the red ball "disappears off to infinity".

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Limits of finite graphs

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This leads us to the Benjamini-Schramm limit.

Limits of finite graphs

Definition: $\mathcal{G}_\bullet$

We define $\mathcal{G}_\bullet$ to be the space of (isomorphism classes of) rooted, connected and locally finite directed graphs, equipped with the projective topology. [BS11, Kai15].

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This space is locally compact and metrizable. And the subsets of $\mathcal{G}_\bullet$ of graphs with bounded vertex degree form compact subsets.

Limits of finite graphs

The Benjamini-Schramm limit

With $\mathcal{G}_\bullet$ so defined, every finite graph G yields a probability measure on $\mathcal{G}_\bullet$ by taking a *random rooting* (G, o) of G.

$$\mu = \frac{1}{|G|} \sum_{v \in G} \delta[(G, o)]$$

That is, we can take a normalized sum of point masses of (isomorphism classes of) the different rootings of G

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We can then say that a sequence of finite graphs G_n *converge in the Benjamini-Schramm sense*, if the μ_n converge (in the weak-$*$ topology) in $\mathcal{M}(\mathcal{G}_\bullet)$ — the space of Borel probability measures on $\mathcal{G}_\bullet$.

Limits of finite graphs

This resolves our earlier problems. Effectively, the Benjamini-Schramm limit captures all of the different conceivable subsequential limits of rootings of G_n, forming a distribution over the limit set of different rootings.

Sidenote: ...what is a graph?

Q: Everyone has their own definition of a "graph"; what definition are we using?

A: For our purposes, a graph will be a pair of vertices and oriented edges, (V, E) with $E \subseteq V \times V$. This would sometimes be called a digraph with no multiple edges. We will also consider edge-labelled graphs, where there is a map $\ell : E \to \mathcal{A}$. Benjamini-Schramm convergence remains the same, we just modify the meaning of $\cong$.

This solves our problem. Now what about these "particular sequences of graphs"?

Rauzy graphs

What about these "Rauzy graphs"

We will start with *Rauzy graphs*, because these are more closely related to the origin of this project.

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We will start by talking about *subshifts*.

Subshifts

Given a finite *alphabet* $\mathcal{A}$ with $|\mathcal{A}| \geq 2$, we can equip all $\mathcal{A}^n$ with the product-of-discrete topology, and then we define the space of (singly) infinite words with the projective topology

$$\mathcal{A}^{\mathbb{N}} := \varprojlim \mathcal{A}^n$$

this space is compact, metrizable, and totally disconnected; we call it the space of infinite words. It comes with a continuous self-map, S, *the shift*.

$$S : \omega_0\omega_1\omega_2 \cdots \mapsto \omega_1\omega_2\omega_3 \ldots$$

We call compact S-closed subsets $X \subseteq \mathcal{A}^{\mathbb{N}}$ *subshifts*.

Languages

As the infinite words are themselves (projective) limits of finite words, it should be unsurprising that the subshifts X are determined by the permitted finite subwords [LM95]. For any subshift X, we can define *the language*, $L(X) \subset \mathcal{A}^*$ (the free monoid on $\mathcal{A}$) to be the set of all finite length words which appear in some infinite word in X. *In fact, it is usually easiest to define a particular subshift by defining the language.*

Example: the golden mean shift

We can define *the golden mean shift* this way, by defining X to be the collection of infinite words which do not contain the subword 11. So the language of X is then:

$$L(X) = \{\epsilon, 0, 1, 00, 01, 10, 001, 010, 100, 101, 0000, \dots\}$$

We can also define $L_n(X) = L(X) \cap \mathcal{A}^n$ to be the subwords of length n. In the above example, $L_3(X) = \{000, 001, 010, 100, 101\}$

Rauzy graph

With these definitions, *the nth Rauzy graph of a subshift* is a graph where the vertices are the length n subwords, and there is an (oriented) edge (u, v) between two length n words if u precedes v in a word w of length $n+1$. That is, $\mathcal{R}^n(X) = (L_n(X), E)$ where

$$(u, v) \text{ in } E \quad \Leftrightarrow \quad \overbrace{w_1\, w_2 \dots w_n}^{u} w_{n+1}, \; w \in L_{n+1}(\omega) \quad \text{with } \underbrace{w_2 \dots w_n w_{n+1}}_{v}$$

Rauzy graphs

Labelled Rauzy graphs

We can also add labels to the edges, by "coloring" the edge with the newly added letter of $\mathcal{A}$. For every edge (u, v),

$$(u, v) \text{ in } E \quad \Leftrightarrow \quad \overbrace{w_1 \, w_2 \ldots w_n}^{u} w_{n+1}, \; w \in L_{n+1}(\omega)$$

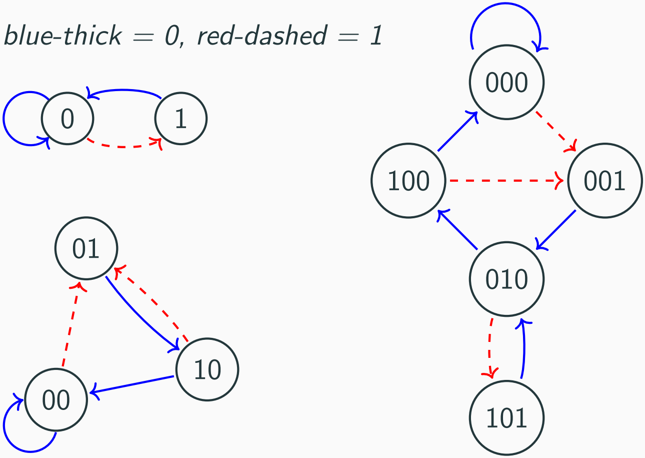
with $v = w_2 \ldots w_n w_{n+1}$,

and $\ell((u, v)) = w_{n+1}$.

We will denote these edge-labelled Rauzy graphs by $\overrightarrow{\mathcal{R}}^n(X_\omega)$.

Rauzy graphs of the golden mean shift

labels: blue-thick = 0, red-dashed = 1



First three Rauzy graphs of the golden mean shift.

The high-complexity case

The golden mean shift, and *the full shift* $\{0, 1\}^{\mathbb{N}}$ are both examples of *shifts of finite type*, where the shifts are determined by a finite number of forbidden symbols. For the golden mean shift, $\{11\}$ is forbidden, and for the full shift, nothing ($\emptyset$). These subshifts are typically of *high-complexity*, having exponential growth in $|L_n(X)|$.

The only exceptions are degenerate cases where $|X| < \infty$, like when X is a periodic shift. This happens if $\{00, 11\}$ are forbidden, or if $\{1\}$ is forbidden, for example.

Rauzy graphs

The high-complexity case

In the case of high-complexity shifts, the Benjamini-Schramm limit of the associated Rauzy graphs has already been studied.

It was shown, for instance, that the labelled Benjamini-Schramm limit of the Rauzy graphs of the full shift (these are known as the *de Bruijn graphs*) converge to

$$\mathfrak{C}ay\left(\mathcal{L}_2, \{\rightarrow, \text{flip} \rightarrow\}\right)$$

where $\mathcal{L}_2$ is *the lamplighter group*, $\mathbb{Z} \wr \mathbb{Z}_2$. The unlabelled graph limit yields the famous Diestel-Leader graph, $DL(2,2)$.

See [Lee16, GLN16, Kai18], and also unpublished work by Kaimanovich, Leeman, and Nagnibeda.

The low-complexity case

The low-complexity case

The low-complexity case, however, is somewhat different, and that is what this thesis addresses.

Low-complexity word

Given an infinite word $\omega \in \mathcal{A}^{\mathbb{N}}$, there is a smallest subshift $X_\omega \subseteq \mathcal{A}^{\mathbb{N}}$ containing ω, which is easily seen to be

$$X_\omega = \overline{\{\ S^k\omega \ : \ k \in \mathbb{N}\ \}}$$

that is, the closure of the orbit of ω is a subshift (recall that S is the shift). We can say that ω is of low-complexity if for some K

$$\limsup_n \frac{|L_n(X_\omega)|}{n} < K$$

The low-complexity case

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That is, low-complexity words have linearly many subwords of a given length, rather than exponentially many!

The low-complexity case

A result of Cassaigne & Frid, [Fri01, Cas96]

Cassaigne showed that the linear growth of subword complexity is equivalent to bounded differences between $|L_{n+1}(X_\omega)|$ and $|L_n(X_\omega)|$. That is, ω is of low-complexity if and only if there is a k such that

$$\forall n.\ |L_{n+1}(X_\omega)| - |L_n(X_\omega)| < k$$

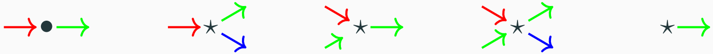
Frid interpreted this in terms of Rauzy graphs, and pointed out that this means that in any Rauzy graph, there must be a bounded number of vertices with in-degree or out-degree greater than 1.

Our first result

Unlabelled convergence

Unlabelled convergence for aperiodic low-complexity words

By the result of Cassaigne & Frid, there are a bounded number of *special vertices* in any $\mathcal{R}^n(X_\omega)$: A special vertex is just a vertex where the in-degree or out-degree are not equal to 1.



$(\bullet)$ *is regular; the* $(\star)$*s are special.*

Unlabelled convergence

Theorem 1

If ω is a low-complexity aperiodic word, then the unlabelled Rauzy graphs $\mathcal{R}^n(X_\omega)$ converge to a point mass concentrated on the line graph in $\mathcal{G}_\bullet$.

Proof sketch:

If ω is aperiodic then $|L_n(X_\omega)| \to \infty$, and so the special vertices can be made to occupy an arbitrarily small part of the graphs. Adding into this the bounded vertex degree of Rauzy graphs (the vertex degree is always less than $2|\mathcal{A}|$), and using that in **any** Rauzy graph $\mathcal{R}^n(X_\omega)$ that there are at most K special vertices for some K, one gets that there are at most $K + (K|\mathcal{A}| - 1)r$ vertices within distance r of a special vertex.

Unlabelled convergence

Proof sketch (cont.):

Since $|L_n(X_\omega)| \to \infty$, this means that for any $r \in \mathbb{N}$, an r-neighbourhood of (uniformly) randomly chosen vertex in $\mathcal{R}^n(X_\omega)$ will not include any special vertex with probability at least

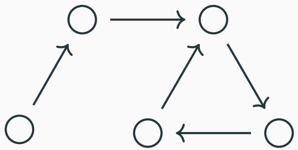
$$1 - \frac{K + (K|\mathcal{A}| - 1)r}{|L_n(X_\omega)|} \to 1$$

and since the (connected) r-neighborhood contains no special vertices, it must be a line[1]. Since every neighborhood randomly converges to a line, and since $\mathcal{G}_\bullet$ is equipped with the projective topology, the Benjamini-Schramm limit is shown to be the point mass of the line graph. $\square$

[1] It cannot be a cycle, or else X_ω would be finite.

Tidying up: the finite cases

If ω is not aperiodic, then it is either eventually periodic or periodic. In these two cases, the Rauzy graphs both stabilize to either a finite cycle (the periodic case) or else a graph that resembles:



The cycle has only one symmetry class, so the Benjamini-Schramm limit is just a point pass on the finite cycle. In the eventually periodic case, the above graph is rigid, so the limit is a uniform measure on all distinct rootings of the graph above (the length of the "handle" and the size of the "loop" may be different).

Unlabelled convergence

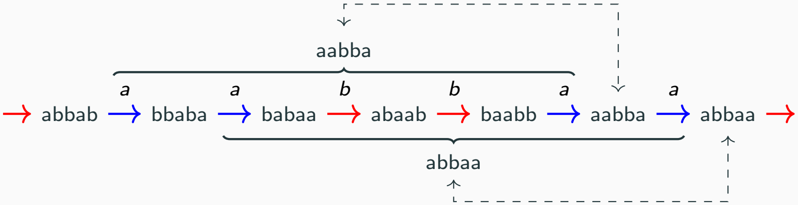
This classifies the unlabelled limits. What about limits of labelled graphs?

The labelled case

The labelled case

Sequences of edges encode finite subwords

For labelled Rauzy graphs, the labels of subsequent edges encode a sequence of letters $w_{n+1}w_{n+2}\dots w_{n+k}$, which are themselves a word in $L(X_\omega)$. Since (by theorem 1) typical neighbourhoods are lines, this means that when we randomly sample a neighbourhood in the Rauzy graphs, we are really sampling finite words from ω.



The labelled case

This leads us to ask, how do we make sense of the “probability” or “frequency” of a subword in ω?

Uniform frequencies

It turns out that a finite word u only has well defined frequencies if the following limit exists *uniformly in k* [2]:

$$\begin{aligned}\mathrm{freq}_u(S^k\omega) &:= \lim_n \frac{1}{n+1} \#\{ \text{ occurrences of } u \text{ in } \omega_k\omega_{k+1}\ldots\omega_{k+n} \ \} \\ &= \ldots \\ &= \lim_n \frac{1}{n+1}\sum_{i=0}^{n} I_u\left(S^{i+k}\omega\right)\end{aligned}$$

[2] *So require that for some c that* $\lim_n \sup_k \left\| \frac{1}{n+1}\sum_{i=0}^{n} I_u\left(S^{i+k}\omega\right) - c\right\| = 0$

The labelled case

That is, it can be viewed as an *ergodic average*. Moreover, if the frequency of every subword is defined (we need it to be.), then since the I_u functions generate a dense subalgebra[3] of $C(\mathcal{A}^{\mathbb{N}}, \mathbb{R})$, Oxtoby's uniform ergodic theorem ([Oxt52]) gives us that (X_ω, S) is *uniquely ergodic*, and we get that for the unique S-invariant measure μ,

$$\mathrm{freq}_u(\omega) = \mu(u)$$

[3] The span of $\{I_u\}_{u \in \mathcal{A}^*}$ separate points and contain the constant functions.

The case of words with uniform frequencies

In the case of words ω with uniform frequencies, we can identify the Benjamini-Schramm limit using the measure μ on X_ω. We start with the aperiodic case.

Theorem 2

If ω is a low-complexity aperiodic word with uniform frequencies, then the labelled Rauzy graphs $\overrightarrow{\mathcal{R}}^n(X_\omega)$ converge to μ—viewing μ as a distribution on $\mathcal{A}$-configurations of the (bi-infinite) line graph in $\overrightarrow{\mathcal{G}}_\bullet$.

The labelled case

Proof sketch:

① If ω is aperiodic, then following the proof of Theorem 1 we get that for any fixed $\phi > 0$ and size k, for sufficiently large n, a random neighbourhood of diameter k in $\vec{\mathcal{R}}^n(X_\omega)$ resembles a line with probability at least $1 - \phi$.

② By the definition of uniform frequencies, for any u and error $\epsilon > 0$, for sufficiently large diameter k, the frequency of u in $\omega_i \omega_{i+1} \ldots \omega_{i+k}$ is within ϵ of the true frequency of u, $\mu(u)$.

Combining ① and ②, we can get that at least a $1 - \phi$ proportion of random k-diameter neighbourhoods are lines, and the frequency of u within these lines can be made within ϵ of $\mu(u)$. So the frequency of u in a large Rauzy graph can be bound between $(1-\phi)(1-\epsilon)\mu(u)$ and $(1+\epsilon)\mu(u)$.

The labelled case

The finite case

In the case of periodic or eventually periodic ω (which automatically have uniform frequencies), we can also identify the Benjamini-Schramm subsequential limits, however the limit only exists in the periodic (= *minimal*) case; in the eventually periodic case, there are p subsequential limits where $S^k\omega$ is periodic with period p.

The labelled case

This happens because the labels on the “handle” (pictured below) cycle with n in $\overrightarrow{\mathcal{R}}^n(X\omega)$, but all the graphs are rigid.

$$
\begin{tikzcd}
 & cca \arrow[rr, "b"] & & cab \arrow[dr, "c"] & \\
ccc \arrow[ur, "a"] & & bca \arrow[ur, "b"] & & abc \arrow[ll, "a"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & ccab \arrow[r, "c"] & cabc \arrow[dr, "a"] & \\
ccca \arrow[ur, "b"] & bcab \arrow[ur, "c"] & & abca \arrow[ll, "b"']
\end{tikzcd}
$$

$$\omega = ccc(abc)^\infty$$

In the periodic case, the (unlabelled) graphs only have one symmetry class, so this doesn’t occur, and all Rauzy graphs (for $n > p$) are isomorphic. In this case, the measure is just determined by the chosen (from $\{1, \ldots, p\}$) vertex.

Remark

All of the results of the earlier theorems also apply to *bi-infinite words* $\omega \in \mathcal{A}^{\mathbb{Z}}$. The proofs are unaffected—one simply has to appropriately modify a few definitions.

The non-uniquely-ergodic case

The non-uniquely-ergodic case

The general situation is a mess.

By Theorem 1 for unlabelled Rauzy graphs, we know that any subsequential Benajmini-Schramm limit of $\left(\overrightarrow{\mathcal{R}}^n(X_\omega)\right)_n$ is supported on what is basically a set of $\mathcal{A}$-configurations of the bi-infinite line graph. The difference is that now, the space of S-invariant measures on X_ω is not a singleton, $\{\mu\}$, but some sort of (Choquet) simplex $\mathcal{M}(X_\omega, S)$.

[FM10] is a good reference for this theory for low-complexity words.

The non-uniquely-ergodic case

A recent result of Cyr and Kra showed that low-complexity shifts have finitely many ergodic measures [CK19], generalizing older results of Boshernitzan which apply only to *minimal shifts* [Bos85]. Since we then know that $\mathcal{E}(X_\omega, S) \subset \mathcal{M}(X_\omega, S)$, the set of *ergodic* measures, is finite, we can show that (Prop 6.4.2)

$$\mathcal{E}(X_\omega) = \bigcup_{\substack{Y \subset X_\omega, \\ Y \text{ minimal}}} \mathcal{E}(Y, S)$$

But when there are two minimal subsystems, the behaviour can be complicated.

The non-uniquely-ergodic case

Example without a limit

Take $\mathfrak{t} = 01101001\ldots$ to be the *Thue-Morse word*. Where $\sigma : \{0,1\}^* \circlearrowleft$ is the substitution map

$$\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases}$$

we have that $\mathfrak{t} = \lim_n \sigma^n(0)$. Now, define

$$\omega = \mathfrak{t} \times (ab)^\infty$$

The non-uniquely-ergodic case

Example without a limit (cont.)

With $\omega = \mathfrak{t} \times (ab)^\infty$ defined this way, the Rauzy graphs have the following structure

$$\overbrace{\mathfrak{t}_{n-2}\ldots 110a \rightarrow \mathfrak{t}_{n-3}\ldots 10ab \rightarrow \cdots \rightarrow 10aba\ldots \rightarrow 0abab\ldots}^{\textit{size} = n-1,\ \text{labels from } \{a,b\}}$$

$$\begin{tikzcd}
\mathfrak{t}_{n-2}\ldots 110a \arrow[r] & \mathfrak{t}_{n-3}\ldots 10ab \arrow[r] & \cdots \arrow[r] & 10aba\ldots \arrow[r] & 0abab\ldots \arrow[d] \\
\underbrace{\overrightarrow{\mathcal{R}}^n(X_{\mathfrak{t}})}_{3n \le \textit{size} \le \frac{10}{3}n} \arrow[u] & & & & \underbrace{\overrightarrow{\mathcal{R}}^n(X_{(ab)^\infty})}_{\textit{size}\ =\ 2}
\end{tikzcd}$$

With $\left|\overrightarrow{\mathcal{R}}^n(X_{\mathfrak{t}})\right| n^{-1}$ oscillating between 3 and $\frac{10}{3}$, attaining both values as limit points.

Example without a limit (cont.)

One gets that, because the size of $\overrightarrow{\mathcal{R}}^n(X_{\mathrm{t}})$ oscillates in this way, the Benjamini-Schramm limit oscillates along with it. Where μ is the (unique) ergodic measure on X_{t}, and ν the uniform measure on $X_{(ab)^\infty}$, the Benjamini-Schramm limit attains

$$\frac{3\mu + \nu}{4} \text{ and } \frac{10/3\mu + \nu}{13/3}$$

as subsequential limits.

The non-uniquely-ergodic case

Example without a limit (cont.)

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as subsequential limits.

And in fact, all possible subsequential limits are a convex combination of these two measures.

The non-uniquely-ergodic case

Arrow reversal

Now, consider the related word, $\omega' = (ab)^\infty \times \mathfrak{t}$. Then the Rauzy graphs look like

$$\overbrace{\hspace{12cm}}^{\textit{size} = n-1,\ \text{labels from } \{0,1\}}$$

$$\begin{tikzcd}
abab\ldots0 \arrow[r] & aba\ldots01 \arrow[r] & \cdots \arrow[r] & ab01\ldots\mathfrak{t}_{n-3} \arrow[r] & a011\ldots\mathfrak{t}_{n-2} \arrow[d] \\
\underbrace{\overrightarrow{\mathcal{R}}^n(X_{(ab)^\infty})}_{\textit{size}\ =\ 2} \arrow[u] & & & & \underbrace{\overrightarrow{\mathcal{R}}^n(X_{\mathfrak{t}})}_{3n \le \textit{size} \le \frac{10}{3}n}
\end{tikzcd}$$

Here, by contrast, $\{a, b\}$ only appear on two edges. The Benjamini-Schramm limit actually gives μ, the ergodic measure on $(X_{\mathfrak{t}}, S)$!

The non-uniquely-ergodic case

Conclusion

While in the non-uniquely-ergodic case the Benjamini-Schramm limit set can be viewed as a subset of $\mathcal{M}(X_\omega, S)$, the limit set itself does not necessarily contain any ergodic measures (though it can), and it may or may not be a singleton (so the limit may or may not exist).

Future work

The unsatisfying loose end of this is the case of the minimal non-uniquely ergodic case. The constructions provided used non-minimality in order to use growth of distinct sub-Rauzy graphs to compute subsequential limits. For minimal words, where we cannot do this, it is less obvious (to me, at least) what can happen!

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Thank you!